Nonlinear Propagation of a Wave Packet in a Two-Dimensional Lined Duct

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Abstract

THE method of multiple scales is used to analyze the nonlinear effects of the gas motion and the acoustic lining material on the propagation and attenuation of a wave packet in a two-dimensional lined duct of a uniform cross section. The partial differential equations describing the space and time variations of amplitude and phase are obtained and used to show that both the monochromatic waves and the pure amplitude-modulated wave packet are stable in the frequency range of interest. The spatial attenuation vs frequency curve for the pure amplitude-modulated wave packet is the flattest one compared with those of linear and nonlinear monochromatic waves. For the pure amplitude-modulated wave packet, a threshold exists near the resonant frequency below which the nonlinear effect is favorable for spatial attenuation, although adverse for temporal attenuation, and above which the nonlinear effects are reversed for spatial attention and temporal attenuation, respectively.

Contents

The acoustic impedance of the lining material for typical jet engines becomes sound pressure dependent when the sound pressure level exceeds 130 dB (re: 0.0002 dyn/cm^2). All existing nonlinear analyses of strongly dispersive waves in a duct are limited to a monochromatic wave in a lined duct[1-4] or to a wave packet in a hard-walled duct.[5]

The present analysis considers the nonlinear propagation of a sound consisting of a group of waves centered about a frequency in a uniform two-dimensional duct whose walls are lined with a nonlinear acoustic material. The gas is assumed to be inviscid, irrotational, and initially quiescent with a uniform pressure p_0^* and a uniform density ρ_0^*. We introduce dimensionless quantities by using the half-width of the duct d^*, the ambient speed of sound C_0^*, and ρ_0^* as reference quantities. A Cartesian coordinate system is introduced such that the x axis is along the axis of the duct. When the foregoing assumptions are combined with equations of continuity and momentum, we obtain

$$\frac{\phi^2\phi}{\partial t^2}+\frac{\partial}{\partial t}(\nabla\phi)^2+\frac{1}{2}(\nabla\phi\cdot\nabla)(\nabla\phi)^2$$
$$=\left\{1+(1-\gamma)\left[\frac{\partial\phi}{\partial t}+\frac{1}{2}(\nabla\phi)^2\right]\right\}\nabla^2\phi \tag{1}$$

where ϕ is a velocity potential function, γ is the gas specific heat ratio, and t is the nondimensional time.

The boundary condition at the interface of the duct and the perforated plate backed by honeycomb cavities has been derived in Refs. 3 and 4 and is expressed as (at $y=1$)

$$-\frac{\partial\phi}{\partial t}-\frac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]+\frac{1}{2}\left(\frac{\partial\phi}{\partial t}\right)^2$$
$$+\frac{1}{2}\frac{\partial\phi}{\partial t}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]-\frac{1}{6}(2-\gamma)\left(\frac{\partial\phi}{\partial t}\right)^3$$
$$=\left\{R_0+R_2\left(\frac{\partial\phi}{\partial y}\right)^2+\left[\chi_0+\chi_2\left(\frac{\partial\phi}{\partial y}\right)^2\right]\frac{\partial}{\partial t}\right\}\frac{\partial\phi}{\partial y}+0(\phi^4) \tag{2}$$

where R_0 and R_2 are, respectively, the linear resistance of the liner, and χ_0 and χ_2 are, respectively, the linear and nonlinear reactance. The propagation and the distortion of the waves take place over different length scales and time scales. Hence, the method of multiple scales[5] is used to determine an approximate solution of symmetric mode for Eq. (1) subject to the boundary condition [Eq. (2)]. The solution is sought from

$$\phi(x,y,t)=\sum_{n=1}^{3}\epsilon^n\phi_n(y,X_0,X_1,X_2,T_0,T_1,T_2)+0(\epsilon^4) \tag{3}$$

where the dimensionless parameter ϵ characterizes the amplitude of the wave, X_0 is a short scale characterizing the wavelength, and $X_1=\epsilon x$ and $X_2=\epsilon^2 x$ are long scales characterizing the amplitude and phase modulations with axial distance. T_0 is a fast scale characterizing the period of wave, and $T_1=\epsilon t$ and $T_2=\epsilon^2 t$ are slow scales characterizing the temporal amplitude and phase modulations. Substituting Eq. (3) and the chain rule expressions for both the spatial and temporal derivatives into Eqs. (1) and (2) and equating coefficient of like powers of ϵ, gives (for $n=1,2,3$)

$$\mathcal{L}(\phi_n)\equiv\left(\frac{\partial^2}{\partial X_0^2}+\frac{\partial^2}{\partial y^2}-\frac{\partial^2}{\partial T_0^2}\right)\phi_n=F_{n-1} \tag{4}$$

$$B(\phi_n)\equiv\left(\frac{\partial}{\partial T_0}+R_0\frac{\partial}{\partial y}+\chi_0\frac{\partial^2}{\partial y\partial T_0}\right)\phi_n=G_{n-1}+\tilde{G}_{n-1} \tag{5}$$

where F_{n-1} and $\tilde{G}_{n-1}$ are, respectively, defined in Refs. 3-5, and

$$G_1=-\frac{\partial\phi_1}{\partial T_1}-\chi_0\frac{\partial^2\phi_1}{\partial y\partial T_1},\quad G_2=-\frac{\partial\phi_2}{\partial T_1}-\frac{\partial\phi_1}{\partial T_2}$$
$$-\frac{\partial\phi_1}{\partial X_0}\frac{\partial\phi_1}{\partial X_1}+\frac{\partial\phi_1}{\partial T_2}\frac{\partial\phi_1}{\partial T_1}-\chi_0\left[\frac{\partial^2\phi_1}{\partial T_2\partial y}+\frac{\partial^2\phi_2}{\partial T_1\partial y}\right]$$

When $n=1$, the solution of Eqs. (4) and (5) is taken to be

$$\phi_1=A(X_1,X_2,T_1,T_2)\cos\kappa y e^{i(kX_0-\omega T_0)}+\text{complex conj.} \tag{6}$$

When $n=2$ and 3, the inhomogeneous parts of Eqs. (4) and (5) contain the $e^{i(kx_0-\omega T_0)}$ term which is the homogeneous solution of the first-order problem. Thus the second- and third-order problems have solutions if and only if the inhomogeneous parts are orthogonal to every solution of the adjoint homogeneous problem. This solvability condition leads to two equations describing the variations of amplitude

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a and phase β with space and time; and they are

$$\frac{\partial a}{\partial x}+\alpha_0 a+K_I\frac{\partial a}{\partial t}-K_2 a\frac{\partial \beta}{\partial t}+C_r\left[\frac{\partial^2 a}{\partial t^2}-\left(\frac{\partial \beta}{\partial t}\right)^2 a\right]$$
$$-C_I\left(2\frac{\partial \beta}{\partial t}\frac{\partial a}{\partial t}+a\frac{\partial^2 \beta}{\partial t^2}\right)=-\alpha_2 a^3\epsilon^2 \qquad (7)$$

$$\frac{\partial \beta}{\partial x}+K_I\frac{\partial \beta}{\partial t}+K_2\frac{\partial a}{a\partial t}+C_r\left(2\frac{\partial \beta}{a\partial t}\frac{\partial a}{\partial t}+\frac{\partial^2 \beta}{\partial t^2}\right)$$
$$+C_I\left[\frac{\partial^2 a}{a\partial t^2}-\left(\frac{\partial \beta}{\partial t}\right)^2\right]=-k_2 a^2\epsilon^2 \qquad (8)$$

where α_2 and k_2 are defined in Refs. 3 and 4, C_r and C_I are defined in the full paper, and

$$K_I+iK_2=C_{xt}=\frac{\omega}{k}\left[I+\frac{2i\kappa(\cos\kappa-\chi_0\kappa\sin\kappa)\cos\kappa}{\omega(2\kappa+\sin 2\kappa)(R_0-i\omega\chi_0)}\right] \qquad (9)$$

By assuming that all variables are independent of X_I and T_I, we are able to investigate the propagation and attenuation of monochromatic waves and pure amplitude modulated waves.

A. The Case of Monochromatic Waves

The amplitude and phase monochromatic waves depend on x only and Eq. (7) can be simplified and reduced to Eq. (42) in Ref. 4.

The attenuation spectra are shown in Fig. 1 in which h is the depth of cavities; b,Ω, and σ_0 are the thickness, porosity, and linear resistivity of the thin perforated plate, respectively; and S is the structure factor.

The new independent variables $\zeta=x-K_I t$ and $\eta=x$, which represent a frame of coordinate system moving with the linear group velocity K_I, are introduced into Eqs. (7) and (8), then the nonlinear Schrödinger equations are obtained and used to determine the stability of waves. To begin with, we let

$$a=a_0(\eta)+\tilde{a}_I e^{i(k\eta-\omega\zeta)},\quad \beta=\beta_0(\eta)+\tilde{\beta}_I e^{i(k\eta-\omega\zeta)} \qquad (10)$$

where the second terms on the right-hand side are small compared with the preceding terms, and $\tilde{a}_I$ and $\tilde{\beta}_I$ are constant. Substituting Eq. (10) into the nonlinear Schrödinger equations and collecting those terms of which the magnitude of order are $O(\tilde{a}_I)$ or $O(\tilde{\beta}_I)$, we obtain

$$2k=i[\alpha_0+3\epsilon^2a^2\alpha_2]\pm[-(\alpha_0+3\epsilon^2a^2\alpha_2)^2$$
$$-4\omega^2k_2^2-4i\omega NK_2k_2\epsilon^2a^2]^{1/2} \qquad (12)$$

where $N=2$ and ω is a real value. The positive sign between two brackets in Eq. (11) is chosen so that a non-trivial solution is obtained for $k_2=0$. Figure 2 shows that the imaginary part of k is positive and the monochromatic waves will decay exponentially in the η direction and are stable for the frequencies considered.

B. The Case of Pure Amplitude-Modulated Wave Packet

Phase β is constant and Eqs. (7) and (8) are simplified to

$$\frac{\partial a}{\partial x}=-a\left\{\alpha_0+\epsilon^2a^2\alpha_2\left(I-\frac{K_I}{K_2}\frac{k_2}{\alpha_2}\right)\right\},\quad \frac{\partial a}{\partial t}=-\epsilon^2a^3\frac{k_2}{K_2} \qquad (12)$$

The same lining material and duct for case A are used here and the result is shown in Fig. 1. The spatial attenuation vs frequency curve for the pure amplitude-modulated wave packet is the flattest one compared with those of monochromatic and linear waves. The adverse nonlinear effect for a wave packet is limited to those frequencies higher than the threshold, which is somewhat below resonant frequency. Figure 3 shows that the nonlinear effect due to spatial attenuation is favorable for those frequencies higher than the threshold mentioned earlier and is adverse for those frequencies below the threshold.

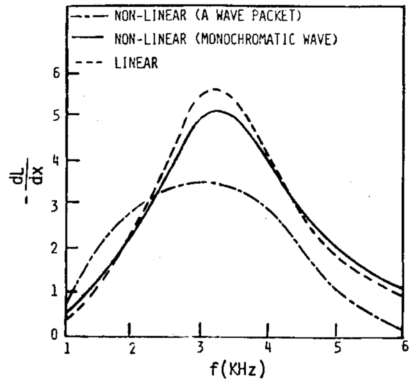

Fig. 1 Variation of the absorption coefficient of the lowest mode with frequency. $d^*=1$ in.; $b^*=0.015$ in.; $h^*=0.75$ in.; $\Omega=0.95$; $S=1.15$; $\sigma_0=53.33$; $R_2=12164$; $|\partial\phi/\partial y|=0.0024$ at the duct/liner interface.

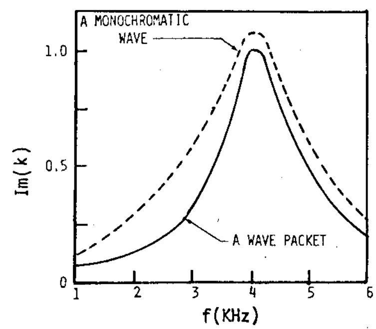

Fig. 2 Variation of the imaginary part of k with frequency for stability analysis (same lined duct and particle velocity at wall as in Fig. 1).

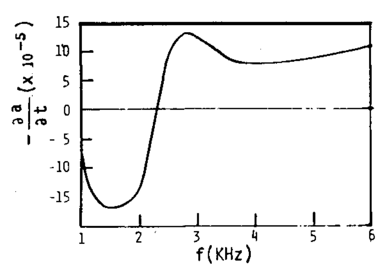

Fig. 3 Variation of temporal attenuation of the lowest mode with frequency (same lined duct and particle velocity at wall as in Fig. 1).

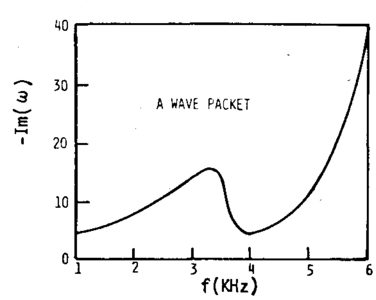

Fig. 4 Variation of the imaginary part of ω with frequency for stability analysis (same lined duct and particle velocity at wall as in Fig. 1).

The same stability analysis shown in case A is used here and we arrive at the same equation as Eq. (11) except $N=3$ and ω is a complex value. Figures 2 and 4 show that the imaginary parts of the k and ω are positive and negative, respectively. Therefore, the amplitude of the pure amplitude-modulated wave packet will decay exponentially in both ζ and η directions and the wave packet is stable for the frequency range considered.

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